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Some Applications of Completely Monotonic Functions

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Abstract

In this article, we introduce some applications of completely monotonic functions in the theory of special functions, probability and statistics theory, and physics.

Keywords: Completely monotonic function; Gamma function; Digamma function; distribution function

1. Introduction

We first recall some definitions and basic results.

Definition 1 ([35]). A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in N_0$

$$(-1)^n f^{(n)}(x) \geq 0, \ x \in I^o.$$

Some mathematicians use the terminology *completely monotone* instead of *completely monotonic*.

Here in Definition 1, and throughout the paper, N_0 is the set of all nonnegative integers; I^o is the interior of the interval I and $C(I)$ is the set of all continuous functions on I. Also we use N to denote the set of all positive integers.

If the functions f and g are both completely monotonic on the interval I, then, by definition, the functions $f + g$ and $f \cdot g$ are also completely monotonic on I.

For the representations of the completely monotonic functions on $(0, \infty)$ or $[0, \infty)$, the following two results are the well-known Bernstein's Theorems (see [35, Chapter IV, Section 12]).

Theorem 2. The function f is completely monotonic on the interval $(0, \infty)$ if and only if there exists an increasing function $\alpha(t)$ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt} \, d\alpha(t).$$

Theorem 3. The function f is completely monotonic on the interval $[0, \infty)$ if and only if there exists a bounded and increasing function $\alpha(t)$ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t).$$

There is rich literature on completely monotonic functions. For more recent works, see, for example, [1, 2, 5–19, 22, 25–27, 30–34, 36, 37].

2. Some applications

The completely monotonic functions have remarkable applications in the theory of special functions, probability and statistics theories, physics, etc. For example, Muldoon [23] proved

Theorem 4 ([23]). Suppose that $f > 0$ on the interval $I = (a, \infty)$ with $a \geq 0$ and that

$$(\ln f)'' \in CM(I).$$

If there exists a strictly increasing sequence $\{x_k\}_1^\infty \subset I$ such that $\sum_{k=1}^\infty (1/x_k)$ diverges and if

$$f(x_k) = \Gamma(x_k), k \in N,$$

then

$$f(x) = \Gamma(x),\ x \in I.$$

Here Γ denotes the gamma function. Please note that the case, in which $\{x_k\}_1^\infty \subset I$ is strictly increasing and bounded, is already included in Theorem 4.

The next result is also proved in [23].

Theorem 5 ([23]). Let $f(x)$ be such that $f'(x) > 0, x \in I$. Here $I = (a, \infty)$ with $a \geq \alpha$, where $\alpha = 1.46163\ldots$ is the unique positive zero of $\psi(x)$. If

$$f/f' \in CM(I)$$

and if there exists a strictly increasing sequence $\{x_k\}_1^\infty \subset I$ such that $\sum_{k=1}^\infty (1/x_k)$ diverges and

$$f(x_k) = \Gamma(x_k), k \in N,$$

then

$$f(x) = \Gamma(x), x \in I.$$

Here, in Theorem 5, ψ denotes the psi or digamma function.

Kimberling [20] gave some applications of completely monotonic functions in probability and statistics.

Theorem 6 ([20]). Suppose that f is a strictly decreasing function from $[0, \infty)$ into $[0, 1]$, that $f(0) = 1$ and $f(\infty) \geq 0$, and that $\{F_n\}$ is a sequence of continuous distribution functions over which f is admissible. Then f is completely monotonic on $[0, \infty)$.

Please note that in Theorem 6, $f(\infty)$ is the limit of $f(x)$ as $x \to \infty$.

Theorem 7 ([20]). Suppose that $\{F_n\}$ is a sequence of distribution functions. Also suppose that f is function from $[0, \infty]$ onto $[0, 1]$ and that f is completely monotonic on $[0, \infty)$. Then f is admissible over $\{F_n\}$.

Here, in Theorems 6 and 7, that f is admissible over $\{F_n\}$ means that there exists a probability space (Ω, A, P) and a sequence $\{X_n\}$ of random variables defined on that space such that F_n is the distribution function of $\{X_n\}$ for $n \in N$ and $\{X_n\}$ is exchangeable under f. For meanings of other terminologies here see [20]. Also see [21, 24, 28, 29] for more applications in probability and statistics.

For the applications in physics, Day [3] established the following result.

Theorem 8 ([3]). In a linear viscoelastic material the relaxation function $G(x)$ has the property that $G(x) - G(\infty)$ is completely monotonic on $[0, \infty)$ if and only if the work done on retraced paths is increased by delay.

In a non-linear viscoelastic material, the author also proved under certain conditions $G(x)$, the infinitesimal relaxation function, has the property that $G(x) - G(\infty)$ is completely monotonic on $[0, \infty)$ (see [3, Theorem 2] for details of this result). For the meanings of terminologies used here see [3]. The applications in physics can also be found, for example, in [4].

Declaration of Conflicting Interests

The authors declare no potential conflicts of interest with respect to the research, authorship and publication of this article.

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