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Oscillation criteria for second-order neutral differential equations with Mixed Arguments

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Abstract

The objective of this paper, we obtain new criteria for oscillation of the second-order neutral delay differential equations of the form

$$(p(t)[x(t)+q(t)x(\sigma(t)]')'=0\,,t\geq t_0>0$$

few new results are presented that improve related ones., our approach for the positive solutions of studied equations.

Keywords: Second-order, neutral differential equations, delay, oscillation. , Mixed Arguments

1. Introduction

In this paper we shell study the oscillation criteria the second-order neutral delay differential equation

$$(p(t)[x(t)+q(t)x(\sigma(t)]')'=0\,,t\geq t_0>0 \qquad \text{(A)}$$

In this paper, it is assumed that the following conditions hold:

$(\boldsymbol{A_1})$ p(t),q(t), $\in \boldsymbol{C^1}([\boldsymbol{t_0},\infty))$, r(t), $\in \boldsymbol{C}([\boldsymbol{t_0},\infty))$,p(t)$>\mathbf{0}$, $0\leq \boldsymbol{q(t)}\leq \boldsymbol{q_0}<\mathbf{1}$,r(t)$>\mathbf{0}$

$(\boldsymbol{A_2})$ s(t),$\boldsymbol{\sigma}$(t), $\in \boldsymbol{C^1}([\boldsymbol{t_0},\infty))$,$\boldsymbol{\sigma}$(t), $\in \boldsymbol{C}([\boldsymbol{t_0},\infty))$,s(t)$\leq \boldsymbol{t}$, $\boldsymbol{\sigma(t)}\geq \boldsymbol{t}, \boldsymbol{s'(t)}\geq \mathbf{0}$,

$\lim\limits_{t\to\infty} \boldsymbol{s(t)}=\mathbf{0}$,

$\lim\limits_{t\to\infty} \boldsymbol{\sigma(t)}=\mathbf{0}$, Supposed that (A) is in a canonical form

$$\lim_{t\to\infty} \boldsymbol{R(t) = 0}, \qquad (1.1)$$

Where $R(t) = \int_{t_0}^{t} \frac{1}{p(s)} d(s)$

From the Eq. (A) we mean a function $x(t) \in C([T_x, \infty))$, $T_x \geqslant t_0$, which relish the property

$(p(t)[x(t) + q(t)x(\sigma(t))]' \in C^1([T_x, \infty))$ and x(t) satisfies Eq. (A) on

$[T_x, \infty)$. We consider only those solutions x(t) of (A) for which $\sup\{|x(t)|\ t \geqslant T\} > 0$ for all $T \geqslant T_x$.

We assume that (A) possesses such a solution.

The solution x(t) of (A) is said to be oscillatory if it is neither lastly positive or lastly negative. Otherwise, it is said to be non oscillatory.The equation itself oscillatory. if all its solutions oscillate. Then the equation (A) it is sufficient to deal only with positive solutions of (A)..

According to this approach, great many authors have examined the oscillation of neutral differential equations and various method for developing oscillatory results. I used several articles for help [1-19]

Grammatikopoulos et al. [9] established the condition

$$\int_{t_0}^{\infty} r(s)[1 - g(s-\delta)]ds = \infty$$

That insures oscillation of the linear neutral differential equation

$$(x(t) + q(t)x(t-\tau))'' + r(t)x(t-\delta) = 0$$

Grace and Lalli [8] showed that if (1.1) holds and there exists an auxiliary function

$$\theta \in C^1([t_0, \infty), R])$$

such that

$$\int_{t_0}^{\infty} (\theta(s)r(s[1-q(s-\delta)] - \frac{(\theta'(s))' r(s-\delta)}{4\theta(s)}) = \infty$$

then the linear neutral differential equation

$$(p(t)(x(t) + q(t)x(t-\tau))')' + r(t)x(t-\delta) = 0$$

is oscillatory. Ruan in [19] find out oscillation of second-order neutral nonlinear differential equations

$$(a(t)(x(t) + q(t)x(t-\tau))')' + r(t)f(x(t-\sigma)) = 0.$$

Li et al. [8] studied oscillation behavior of non canonical second-order neutral differential equations

$$\left(p(t)[x(t) + r(t)x(S(t))]'\right)' + q(t)x(\sigma(t)) = 0$$

Relating oscillation of these equations to existence of positive solutions to associated first-order functional differential inequalities.

The aim of this paper is to give new oscillatory criteria for (A) by using new monotonic properties for positive solutions of (A) whose oscillatory behavior is known in advance. The first results in this direction for second-order ordinary delay equations were given by Koplatadze [11] and Wang [15] who proved that

$$x''(t) + p(t)x(S(t)) = 0$$

(1.2)

is oscillatory provided that the corresponding first-order differential equation

$$y'(t) + S(t)r(t)y(\tau(t)) = 0$$

is oscillatory, which leads to the following oscillatory criterion for (1.2):

$$\lim_{t\to\infty} inf \int_{p(t)}^{t} S(s)p(s)ds > \frac{1}{e}$$

Parhi and Mohanty [7] investigated the higher-order neutral differential equation

$$[x(t) + q(t)x(\sigma(t))]^{(n)} + r(t)x(S(t)) = 0$$

by comparing it with the first-order delay differential inequality. Namely, for n = 2, they obtained the following oscillatory criterion

$$\lim_{t\to\infty} inf \int_{p(t)}^{t} S(s)(1 - q(\tau(s))r(s)ds > \frac{2}{e}$$

Later, Kusano and Wang [15] used a variant of the Mahfoud's comparison principle [18] and proved that

(1.2) is oscillatory provided that

2. Main results

This section, we establish required oscillatory criteria. All functional inequalities considered in this section are assumed to hold eventually, for every non oscillatory solution x(t) of (A) we establish the corresponding function

$$y(t) = x(t) + q(t)x(\sigma(t))$$

(2.1)

and hence we can rewrite (A) in the form

$$\left(p(t)y'(t)\right)' + p(t)x(S(t)) = 0.$$

(2.2) If $x(t)$ is a positive solution of (A), due to (1.1), it is easy to see that the corresponding function $y(t)$ satisfies

$$y(t) > 0, \quad P(t)y'(t) > 0, \quad \big(P(t)y'(t)\big)' < 0.$$

Lemma 2.1. *If* $x(t)$ *is a positive solution of* (A), *then* $y(t)$ *is a positive solution of the inequality*

$$\big(p(t)y'(t)\big)' + r(t)(1 - q_0)y(S(t)) \leqslant 0. \tag{2.4}$$

Proof. Assume that $x(t)$ is a positive solution of (A), taking into account (2.1), (H_1), (H_2), and (2.3) we have

$$x(t) = y(t) - q(t)x(\sigma(t)) \geqslant y(t) - q(t)y(\sigma(t)) \geqslant y(t) - q(t)y(t) \geqslant y(t)(1 - q_0).$$

That yields $x(S(t)) \geqslant y(S(t))(1 - q_0)$ and in view of (2.2), Eq. (A) can be rewritten in the form

$$\big(p(t)y'(t)\big)' + r(t)(1 - q_0)y(S(t)) \leqslant 0.$$

Lemma 2.2. *If* $y(t)$ *is a positive solution of* (2.4) *such that* $\liminf_{t\to\infty} y(t) > 0$*, then the differential equation*

$$\big(p(t)z'(t)\big)' + r(t)(1 - q_0)z(S(t)) = 0 \tag{2.5}$$

has a positive solution $z(t)$ *and* $\liminf z(t) > 0$.

Declaration of Conflicting Interests

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