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Odd-Even Congruence Labelling of Some Well-Known Graphs

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Abstract

Introduction: In the field of graph theory, graph labelling is one of the fastest growing area. A graph labeling is defined as an assignment of integers to the nodes or edges or both based on certain conditions. Many types of graph labeling techniques have been studied by several authors.

Objectives: The application of labelling of graphs has diverse fields in the study of the data base management, secret sharing schemes, physical cosmology, debug circuit, X-ray, crystallography, astronomy, radar and much more.

Methods: This paper is based on Odd-even congruence labeling of different types of graphs. The vertices and edges are labeled by an assignment of natural numbers. It is based on modular arithmetic property known as congruence graph labelling of a graph. In congruence graph labelling vertices are assigned by odd integers and edges are assigned by even integers with the property of congruence labelling.

Results: This labeling method has been identified on tadpole graph, pan graph, generalized Petersen graph, wheel graph.

Keywords: Labeling, Congruence labeling, Odd-even congruence labeling.

1. Introduction

In this paper all graphs $\boldsymbol{G}$ considered as finite, undirected, connected and without loops. The vertices and edges of a graph $\boldsymbol{G}$ denoted by $\boldsymbol{V}(\boldsymbol{G})$ and $\boldsymbol{E}(\boldsymbol{G})$ respectively. The cardinality of vertex set is denoted by $|\boldsymbol{V}(\boldsymbol{G})|$ and the edge set is denoted by $|\boldsymbol{E}(\boldsymbol{G})|$ are called the order and size of the graph $\boldsymbol{G}$. For standard terminology of Graph Theory we used [1]. For all detailed survey of graph labeling, we refer [2]. While studying graph theory, one that has gained a lot of popularity during the last $\boldsymbol{62}$ years is the concept of labeling of graphs due to its wide range of applications. A labeling of a graph $\boldsymbol{G}$ is one-to-one mapping that carries the set of graph elements onto a set of numbers, called labels. In 1967, Rosa [5] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling methods have been studied by several authors.

Odd-even congruence labeling introduced by G.Thamizhendhi and K.Kanakambika [6] and they proved behaviour of several graphs like bipartite graph, comb graph, star graph, graph acquired by connecting two copies of even cycle $\boldsymbol{C_r}$ by a path $\boldsymbol{P_t}$, shadow graph of the path $\boldsymbol{P_t}$ and the tensor product of $\boldsymbol{K_{1,t}}$ & $\boldsymbol{P_2}$ as Odd-even congruence graph. They defined an odd-even congruence graph, if vertex and edge set are assigned by distinct odd and even integers respectively, further $\boldsymbol{f(u_p) \equiv f(u_q) \left(mod\ g(e)\right)}$, $\boldsymbol{u_p}$ and $\boldsymbol{u_q}$ are adjacent vertices in $\boldsymbol{G}$. In this paper we investigate the existence of Odd-even congruence labeling for tadpole graph, pan graph, generalized petersen graph and wheel graph.

2.Preliminaries

Definition 2.1 [12] A tadpole graph is a type of graph formed by connecting a cycle to a path with a single edge. In other words, it looks like a "head" (the cycle) attached to a "tail" (the path), resembling the shape of a tadpole. If the cycle has n vertices and the path has m vertices, the graph is often denoted as $T(n,m)$.

Definition 2.2 [13] When we join a cyclic graph C_n to a singleton graph K_1 with a bridge then a new graph will be formed known as $n -$ pan graph.

Definition 2.3 The generalized Petersen graph introduced by Watkins. According to Watkins for any integer $n \geq 3$ and $k \in \left\{1,2,\dots\dots.\left\lfloor\frac{(n-1)}{2}\right\rfloor\right\}$, the generalized Petersen graph denoted as $GP(n,k)$ and defined as by the vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and the edge set $E = \{u_i u_{i+k}, u_i v_i, v_i v_{i+1} : 1 \leq i \leq n\}$.

Definition 2.4 [2] A wheel W_n is a graph obtained from a graph with single vertex v and a cycle C_n , by connecting all vertices of C_n to v. The vertex corresponding to v is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges.

3.Main Results

Theorem 3.1 The tadpole graph $T(n,m)$ is an Odd-even congruence graph.
Proof:
Suppose $e_1, e_2, \dots\dots\dots\dots\dots, e_n$ be edges and $v_1, v_2, \dots\dots\dots\dots\dots, v_n$ be vertices of a cycle C_n and $e_{n+2}, e_{n+3}, \dots\dots\dots\dots\dots, e_{n+m}$ be edges and $v_{n+1}, v_{n+2}, \dots\dots\dots\dots\dots, v_{n+m}$ be vertices of path P_m.
To construct tadpole graph $T(n,m)$ join path P_m to the vertex V_n of cycle C_n by an edge e_{n+1}.
Then $\left|V\left(T(n,m)\right)\right| = n + m$
and $\left|E\left(T(n,m)\right)\right| = n + m$
We have $d = \min\{(n+m),(n+m)\}$
$= n + m$
The vertices of $T(n,m)$ are labelled as given below.
Define the bijective function $f_1: V_i \to \{1,3,7,15,\dots\dots\dots\dots\dots\dots\dots..,2^i - 1\}$ $1 \leq i \leq n$
as $f(v_i) = 2^i - 1$
and $g(v_{i+n}) = 2^{i+n} - 1$, $1 \leq i \leq m$
Edges of tadpole are labelled as follows:
$s_1: e_i \to \{2,4,8,\dots\dots\dots\dots\dots\dots\dots..\}$
as $s_1(e_i) = 2^i$
for $1 \leq i \leq n-1$
$s_2(e_n) = |f(v_n) - f(v_1)|$
$s_3(e_{i+n}) = 2^{i+n}$
for $i + n \leq i \leq m$
Clearly $s_1(e_i)$ divides $|f_1(v_i) - f_1(v_{i+1})|$
for $1 \leq i \leq n-1$
$s_2(e_n)$ divides $|f(v_n) - f(v_1)|$
$s_3(e_i)$ divides $|g(u_{i+n}) - g(u_{i+n+1})|$
for $1 \leq i \leq m-1$
Hence the tadpole graph $T(n,m)$ admits an Odd-even congruence graph.

Example: 3.2.
Consider a tadpole graph $T(n, m)$ where $n = 5$ and $m = 2$.

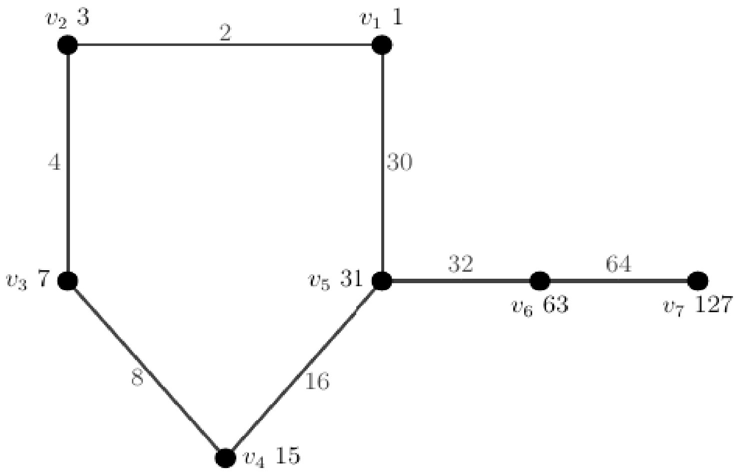

Figure 3.4

Fig:3.1 shows that graph $T(n, m)$ admits odd-even congruence labelling.
Example: 3.3.
Consider a tadpole graph $T(n, m)$ where $n = 6$ and $m = 4$.

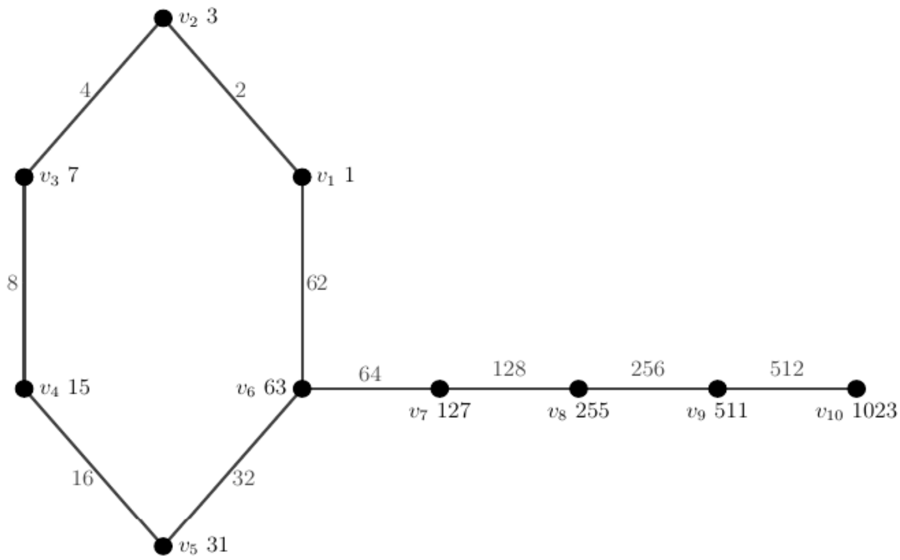

Figure 3.2

Fig:3.2 shows that graph $T(n,m)$ admits odd-even congruence labelling.

Theorem 3.4 The $n-$ pan graph is an Odd-even congruence graph.

Proof:

Suppose $\{v_1, v_2, v_3, \dots \dots \dots \dots \dots \dots \dots \dots \dots v_n,\}$ be the vertex of the cycle graph C_n and $\{e_1, e_2, e_3, \dots \dots \dots \dots \dots \dots \dots \dots \dots\ e_n\}$ where $\{e_i = v_i v_{i+1} : 1 \le i \le n-1\}$ be the edges of the cycle graph C_n. Let v be the vertices of the path of singleton bridge K_1 and let e be the edges of singleton bridge K_1.

Define vertex labelling

$f: v_i \to \{3,7,15, \dots \dots \dots \dots ..\}$ as

$f(v_i) = 2^{i+1} - 1$, $\quad 1 \le i \le n$

$f(v) = 1$

The induced edge labelling

$f^* : e_i \to \{4,8,16, \dots \dots \dots \dots \dots \dots ..\}$ such that

$f^*(e_{i+1}) = 2e_i$, $\qquad 1 \le i \le n-2$

$f^*(e_1) = 4$

$f^*(e) = 2$

$f^*(e_n) = 2f^*(e_{n-1}) - 4$

Now it is clear that $f^*(v_i v_{i+1})$ divides $|f(v_i) - f(v_{i+1})|$ and

$f^*(e)$ divides $|f(v) - f(v_1)|$

Therefore, the $n-$ pan graph admits Odd-even congruence graph.

Example: 3.5.

Consider a $n-$ pan graph where $n = 4$

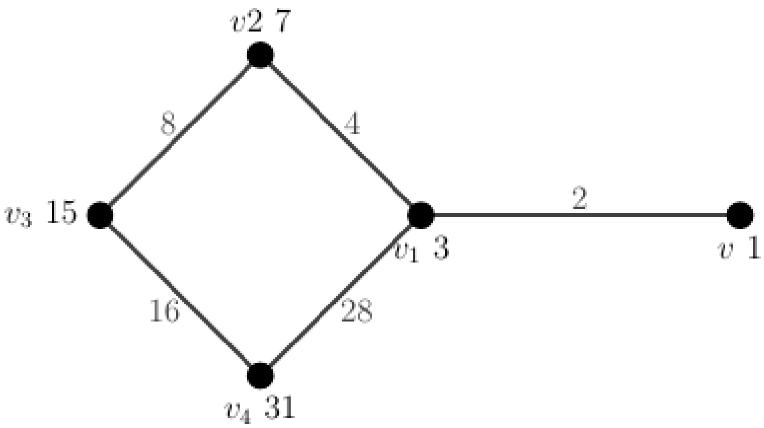

Figure 3.3

Fig:3.3 shows that $n-$pan graph where $n = 4$ admits odd-even congruence labelling.

Example: 3.6.

Consider a $n-$ pan graph where $n = 6$

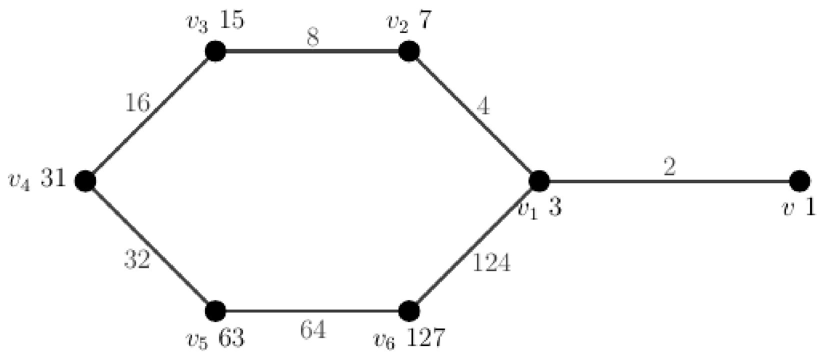

Figure 3.4

Fig:3.4 shows that n −pan graph where $n = 6$ admits odd-even congruence labelling.

Theorem 3.7. Generalized Petersen graph $GP(n, 1)$ is an Odd-even congruence graph.

Proof:

Suppose the vertex set of Petersen graph $GP(n, 1)$ be defined as

$V\big(GP(n, 1)\big) = \{u_i, v_i : 1 \leq i \leq n\}$

and the edge set of Petersen graph $GP(n, 1)$ be defined as

$E\big(GP(n, 1)\big) = \{u_i u_{i+1}, u_1 u_n, u_i v_i, v_i v_{i+1}, v_1 v_n \ : 1 \leq i \leq n-1, 1 \leq j \leq n-1\}$

Where $x_i = u_i u_{i+1}$

$z_i = u_i v_i$

$w_i = v_i v_{i+1}$ respectively.

The vertex labelling defined as

$f: u_i \to \{1,3,5,7, \ldots \ldots \ldots \ldots\}$

Where $f(u_i) = 2^i - 1$ for $1 \leq j \leq n$

$f(v_i) = 2^{i+n} - 1$ for $1 \leq j \leq n$

The edge labelling defined as

$f: x_i \to \{2,4,8,16, \ldots \ldots \ldots \ldots\}$

Where $f(x_i) = 2^i$ for $1 \leq i \leq n-1$

$f(x_n) = f(u_1) - f(u_n)$

6

$f(z_1) = 2f(x_n) + 2$

$f(z_{i+1}) = 2f(z_i)$ for $1 \leq i \leq n-1$

$f(w_1) = f(z_1) + 2$

$f(w_{i+1}) = 2f(w_i)$ for $1 \leq i \leq n-2$

$f(w_n) = f(v_1) - f(v_n)$

It is clear that

$f(x_i)$ divides $|f(u_{i+1}) - f(u_i)|$ for $1 \leq i \leq n-2$

$f(x_n)$ divides $|f(u_1) - f(u_n)|$

$f(z_i)$ divides $|f(u_i) - f(v_i)|$

$f(w_i)$ divides $|f(v_i) - f(v_{i+1})|$ for $1 \leq i \leq n-1$

$f(w_n)$ divides $|f(v_1) - f(v_n)|$

Therefore, we conclude that the generalized Petersen graph $GP(n, 1)$ is an Odd-even congruence graph.

Example: 3.8.

Consider the Generalized Petersen graph $GP(n, 1)$ for $n = 4$

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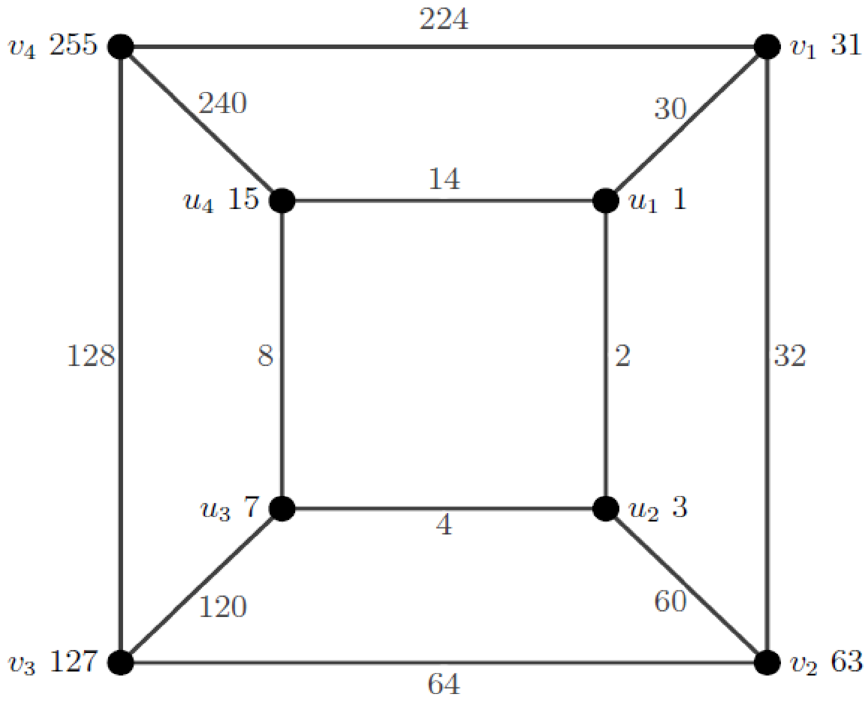

Figure 3.5

Fig:3.5 shows that the Generalized Petersen graph $GP(n, 1)$ for $n = 4$ admits odd-even congruence labelling.

Example: 3.9.

Consider the Generalized Petersen graph $GP(n, 1)$ for $n = 6$

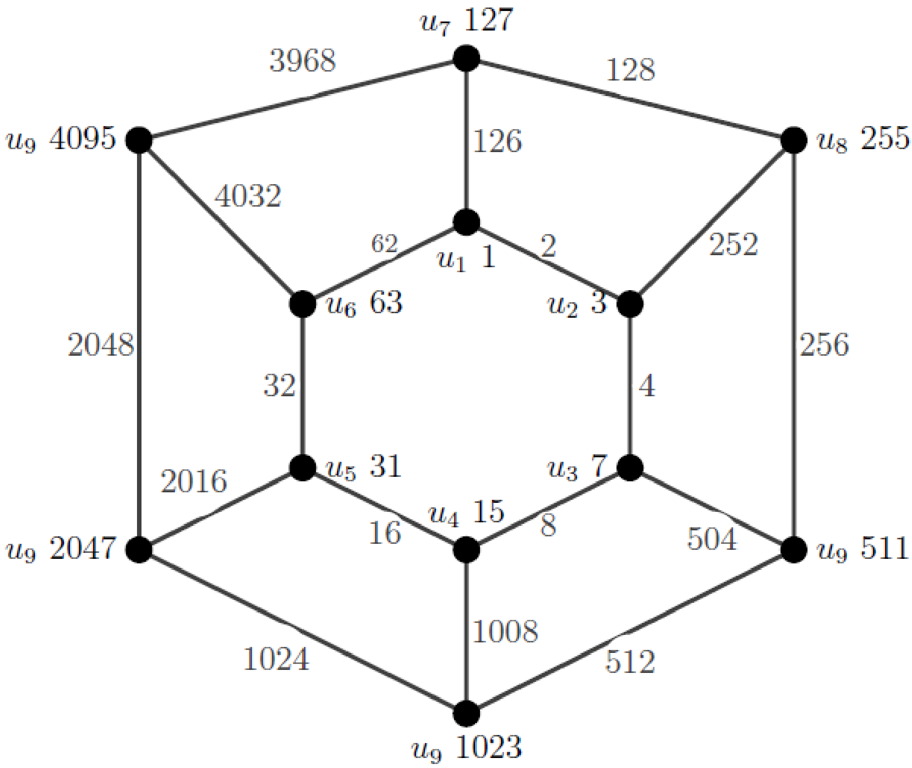

Figure 3.6

Fig:3.6 shows that the Generalized Petersen graph $GP(n,1)$ for $n = 6$ admits odd-even congruence labelling.
Theorem 3.10 The wheel graph W_n is an Odd-even congruence graph.
Proof:
Suppose W_n be the wheel with rim vertices $v_1, v_2, \dots\dots\dots\dots\dots, v_n$ and an apex vertex v. Then $|V\ (W_n)| = n + 1$
and $|E\ (W_n)| = 2n$
We have $d = \min\{(n+1), 2n\}$
$= n + 1$
Label the vertices and edges as follows:
Define the bijective function $f_1: V_i \rightarrow \{3,7,15, \dots\dots\dots\dots\dots\dots\dots..\}$
as $f(v_i) = 2v_i + 1$
for $1 \leq i \leq n$
Where $v_1 = 3$ and $v = 1$
Let the apex vertex v labelled as 1.
Edges of W_n are labelled as
$g: e_i \rightarrow \{4,8,16,32 \dots\dots\dots\dots\dots\dots\dots..\}$
as $g(e_{i+1}) = 2e_i$
for $1 \leq i \leq n-1$
Where $e_1 = 4$
Also $e_i = e_1 - e_{i-1}$ for $i = n$
$h(w_i) = \{2,6,14,30,62, \dots\}$
as $h(w_{i+1}) = 2w_i + 2$
Where $w_1 = 2$
It is clearly that $g(e_i)$ divides $|f(v_i) - f(v_{i+1})|$
for $1 \leq i \leq n-1$

Also $h(w_i)$ divides $|f(v) - f(v_i)|$
$s_3(e_i)$ divides $|g(u_{i+n}) - g(u_{i+n+1})|$
Hence wheel graph admits odd-even congruence labeling.
Example: 3.11.
Consider a wheel graph W_n where $n = 4$.

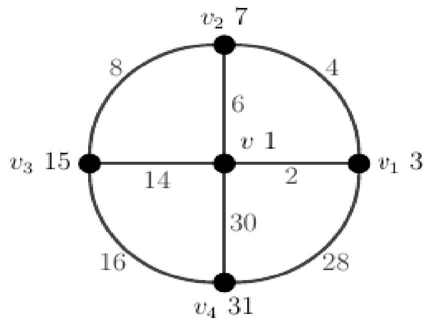

Figure 3.7

Fig:3.7 shows that graph W_n admits odd-even congruence labelling.
Example: 3.12.
Consider a wheel graph W_n where $n = 8$.

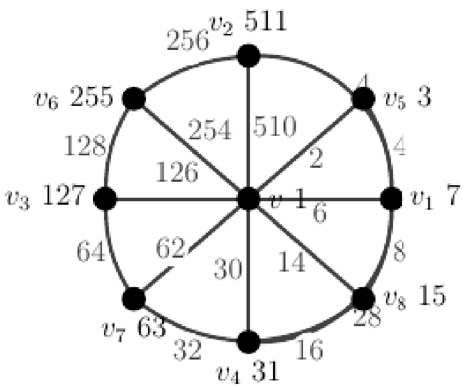

Figure 3.8

Fig:3.8 shows that graph W_n admits odd-even congruence labelling.

Conclusion

The labelling of graphs is an interesting and vast research area which is very useful and it is extended in various topics by several people. In this paper we have studied Odd-even congruence labeling behaviour of tadpole graph, pan graph, generalized Petersen graph, wheel graph. To derive similar results for other graph families is an open problem.

Declaration of Conflicting Interests

The authors declare no potential conflicts of interest with respect to the research, authorship and publication of this article.

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