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Bayawa Transform Method (BTM) for Solving Second Kind Linear Volterra Integro-Differential Equations

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Abstract

In this paper, we exploit Bayawa transform method (BTM) for solving second kind linear Volterra integro-differential equations for verdicts exact solution. The Bayawa Transform Method (BTM) is verified through numerous examples in sequence to show the power of the method.

Keywords: Bayawa Transform Method, Inverse Bayawa Transform, Linear Volterra Integro -differential Equations of the Second Kind, Convolution Theorem

1. Introduction

Volterra integro differential equation showed in numerous physical applications such as nuclear reactors, circuit's analysis, wave propagation, glass forming, nato hydrodymics, visco elasticity, neuron diffusion and biology species coexisting together with increasing and decreasing rates of generating and wind ripples in desert and so on. Volterra integro differential equation contains one or more of the derivatives of unknown function $s(x)$ with respect to x i.e $s'(x)$, $s''(x\ )$, $s'''(x)$, …, outside the integral sign. The Voltera integro differential of the second kind given in the form

$$s^{m}(x) \ = g(x) + \ \lambda \int_0^x k(x,t)\,(s(t))dt \qquad (1)$$

Where $s^{m}(x)$ mth derivative of unknown function $s(x)$ with respect to x. Since (1) contains both operator of differential and integral to attaining the particular solution $s(x)$ of (1), it is necessary to define initial conditions $s'(x)$, $s''(x\ )$, $s'''(x)$, $s^{iv}(x)$, … , $s^{(m-1)}(0)$ by Wazwaz (1997, 2011)

Recently, Bayawa and Haliru (2024) introduced the new integral transform Bayawa transform. Bayawa *et al*

(2025) gave the application of new integral transform Bayawa transform for solving linear Volterra integral equations of the first and the second kinds. Rashdi (2024). Solve the ordinary differential equations with variable coefficient using the new Bayawa transform. Aggarwal *et al* (2018) proposed the application of Aboodh transform for solving liinear Voltera integro -differential of the second kind. Aggarwal and Gupta (2019) discussed the solution of linear Voltera integro differential of the second kind using Kamal transform.

The aim of this research is to apply Bayawa transform method(BTM) for solving linear Volterra integral of the second kind to acquire the exact results.

2. Definition of Bayawa transform

The Bayawa transform of the function $s(t), t \geq 0$ is given by

$$ß\{s(t)\} = S(v) = v^2 \int_0^{\infty} s(t)\, e^{-\frac{t}{v^2}}\, dt, h_1 \leq v \leq\ h_2 \quad (1)$$

2.1 Bayawa transform of the derivatives of functions

(i) $ß[s'(t\ )] = \frac{1}{v^2} S(v) - v^2 s(0)$

(ii) $ß[s''(t\ )] = \frac{1}{v^4} S(v) - s(0) - v^2 s'(0)$

(ii) $ß[s'''(t\ )] = \frac{1}{v^6} S(v) - \frac{1}{v^2} s(0) - s'(0) - v^2 s''(0)$

(iii) $ß[s^n(t\ )]\ = \frac{1}{v^{2m}} S(v) - \sum_{k=0}^{n-1} v^{-2m+2k+4} s^{(k)}(0)$

2.2 Linearity Property theorem for Bayawa transform

Suppose $s_1(t)$ and $s_2(t)$ are Bayawa transform of functions $S_1(v)$ and $S_2(v)$, respectively, then Bayawa transform of $[d s_1(t) + j s_2(t)]$ is given by $[d S_1(v) + j S_2(v)]$, where d and j are arbitrary constants.

2.3 Convolution of two Functions

The Convolution of two functions $s(t)$ and $r(t)$ is denoted by $s(t) * r(t)$ and it is defined by

$$s(t) * r(t) = s * r = \int_0^t s(x)\, r(t-x) dx$$

$$= \int_0^t r(x)\, s(t-x) dx$$

2.4 Convolution theorem for Bayawa transform

Suppose $ß\{s(t)\} = S(v)$ and $ß\{r(t)\} = R(v)$ then $ß\{s(t)\ * r(t)\} =\ ß\{s(t)\}ß\{r(t)\} = \frac{1}{v^2} S(v) R(v)$

2.5 Bayawa transform of some useful functions

Table 1

S/N.	$s(t)$	$ß\{s(t)\} = S(t)$
1.	1	v^4
2.	t	v^6
3.	t^2	$2!\,v^8$
4.	t^3	$3!\,v^{10}$
5.	$t^n, n \in N$	$v^{2n+4}.n!$
6.	e^{at}	$\frac{v^4}{1 - av^2}$
7.	$sinat$	$\frac{av^6}{1 + a^2v^4}$
8.	$cosat$	$\frac{v^4}{1 + a^2v^4}$
9.	$sinhat$	$\frac{av^6}{1 - a^2v^4}$
10.	$coshat$	$\frac{v^4}{1 - a^2v^4}$

2.6 Inverse Bayawa transform

If $ß\{s(t)\} = S(t)$ then $S(t)$ is called the inverse Bayawa transform of $S(t)$ and mathematically it is define as $s(t) = ß^{-1}\{S(t)\}$ where the operator $ß^{-1}$is called the inverse Bayawa transform operator.

2.7 Inverse Bayawa transform of some useful functions

Table 2

S/N.	$S(t)$	$s(t) = ß^{-1}\{S(t)\}$

1.	v^4	1
2.	v^6	t
3.	v^8	$\frac{t^2}{2!}$
4.	v^{10}	$\frac{t^3}{3!}$
5.	v^{2n+4},$n \in N$	$\frac{t^n}{n!}$
6.	$\frac{v^4}{1 - av^2}$	e^{at}
7.	$\frac{av^6}{1+a^2v^4}$	$sinat$
8.	$\frac{v^4}{1+a^2v^4}$	$cosat$
9.	$\frac{av^6}{1 - a^2v^4}$	$sinhat$
10.	$\frac{v^4}{1 - a^2v^4}$	$coshat$

3.1 Main results

3.2 Second kind linear Volterra integro differential equations by Bayawa transform method

Let us assume that the kernel $k(x,t)$ of (1) is a difference kernel that can be expressed as difference $(x-t)$.

$$\left.\begin{array}{r} s^n(x) = g(x) + \beta \int_0^x k(x-t)s(t)dt \\ s(0) = c_0, s'(0) = c_1, \dots, s^{(n-1)}(0) = c_{n-1} \end{array}\right\} \quad (2)$$

Applying the Bayawa transform from both sides of (2), gives

$$\frac{1}{v^{2n}}S(v) - v^{-2n+4}s(0) - v^{-2n+6}s'(0) = ß\{g(x)\} + \beta ß\left\{\int_0^x k(x-t)s(t)dt\right\}$$

$$\frac{1}{v^{2n}}S(v) = v^{-2n+4}s(0) + v^{-2n+6}s'(0) + ß\{g(x)\} + \beta ß\left\{\int_0^x k(x-t)s(t)dt\right\} \quad (3)$$

Using initial conditions $s(0) = c_{0,} s'(0) = c_1, \dots, s^{(n-1)}(0) = c_{n-1}$ equation above, gives

$$\frac{1}{v^{2n}} S(v) = v^{-2n+4}(c_0) + v^{-2n+6}(c_1) + ß\{g(x)\} + \beta ß\left\{\int_0^x k(x-t)s(t)dt\right\} \quad (4)$$

Using convolution theorem of Bayawa transform of (4), gives

$$S(x) = c_0 v^4 + c_1 v^6 + \cdots + c_{n-1} v^n + v^n\, ß\{g(x)\} + \beta v^n ß\{k(x)\}ß\{S(x)\} \quad (5)$$

Taking inverse Bayawa transform of equation above, gives

$$S(v) = c_0 + c_1 x + \cdots + c_{n-1}\frac{x^{n-1}}{n-1} + ß^{-1}\left\{v^n\, ß\{g(x)\}\right\} + \beta ß^{-1}\{v^n ß\{k(x)\}ßS(x)\} \quad (6)$$

3. Applications

Here, we apply Bayawa transform method to solve various second kind linear Volterra integro differential equations. For all illustrations, the exact solutions are getting by Bayawa transform method.

Example1 Consider the second kind linear volterra integro differential equation

$$s'(x) = 1 + \int_0^x s(t)dt, s(0) = 1 \quad (7)$$

Applying the Bayawa transform from both sides of (27), gives

$$ß\{s'(x)\} = ß\{1\} + ß\left\{\int_0^x s(t)dt\right\} \quad (8)$$

Using convolution theorem of Bayawa transform (9), gives

$$\frac{1}{v^2} S(v) - v^2 s(0) = v^4 + \frac{1}{v^2} ß\{1\} * ß\{s(v)\}$$

$$\frac{1}{v^2} S(v) - v^2 s(0) = v^4 + \frac{1}{v^2} \cdot v^4 S(v)$$

$$\frac{1}{v^2} S(v) - v^2 s(0) = v^4 + v^2 S(v) \quad (9)$$

Using initial condition $s(0) = 1$ equation above, gives

$$\frac{1}{v^2} S(v) - v^2(1) = v^4 + v^2 S(v) \quad (10)$$

$$S(v)\left\{\frac{1}{v^2} - v^2\right\} = v^4 + v^2$$

$$S(v) = v^2 \frac{\{v^4 + v^2\}}{1 - v^4}$$

$$S(v) = v^2 \frac{\left\{v^2\{1+v^2\}\right\}}{1 - v^4} \quad (11)$$

Taking inverse Bayawa transform of (12), gives

$$S(v) = ß^{-1}\left\{\frac{v^4}{1-v^2}\right\}$$

Therefore, exact solution $S(x) = e^x$ (12)

Example2 Consider the second kind linear volterra integro-differential equation

$$s'(x) = 2 + \int_0^x s(t)dt, s(0) = 2$$

Applying the Bayawa transform method from both sides of (12) gives

$$ß\{s'(x)\} = ß\{2\} + ß\left\{\int_0^x s(t)dt\right\} \quad (13)$$

Using convolution theorem of Bayawa transform of (13) gives

$$\frac{1}{v^2}S(v) - v^2 s(0) = 2v^4 + \frac{1}{v^2}ß\{1\} * ß\{s(v)\}$$

$$\frac{1}{v^2}S(v) - v^2 s(0) = 2v^4 + \frac{1}{v^2}\cdot v^4 S(v)$$

$$\frac{1}{v^2}S(v) - v^2 s(0) = 2v^4 + v^2 S(v) \quad (14)$$

Using initial condition $s(0) = 2$ equation above

$$\frac{1}{v^2}S(v) - v^2(2) = 2v^4 + v^2 S(v) \quad (15)$$

$$S(v)\left\{\frac{1}{v^2} - v^2\right\} = 2v^4 + 2v^2$$

$$S(v) = 2\frac{\{v^6 + v^4\}}{1 - v^4}$$

$$S(v) = 2\frac{\{v^4\{1+v^2\}\}}{1 - v^4} \quad (16)$$

Taking inverse Bayawa transform of (17) gives

$$S(v) = 2ß^{-1}\left\{\frac{v^4}{1-v^2}\right\}$$

Therefore, exact solution $S(x) = 2e^x$ (17)

Example3 Consider the second kind linear volterra integro-differential equation

$$s''(x) = x + \int_0^x k(x-t)s(t)dt, s(0) = 0, s'(0) = 1 \tag{18}$$

Applying the Bayawa transform from both sides of (19) gives

$$ß\{s''(x)\} = ß\{x\} + ß\left\{\int_0^x k(x-t)s(t)dt\right\} \tag{19}$$

Using convolution theorem of Bayawa transform of (19) gives

$$\frac{1}{v^4}S(v) - s(0) - v^2 s'(0) = v^6 + \frac{1}{v^2}ß\{x\} * ß\{s(v)\}$$

$$\frac{1}{v^4}S(v) - s(0) - v^2 s'(0) = v^6 + v^4 S(v) \tag{20}$$

Using initial condition s $s(0) = 0, s'(0) = 1$ equation above

$$\frac{1}{v^4}S(v) - 0 - v^2(1) = v^6 + v^2 \tag{21}$$

$$S(v)\left\{\frac{1}{v^4} - v^4\right\} = v^6 + v^2$$

$$S(v) = v^4 \frac{\{v^6+v^2\}}{1-v^8} \tag{22}$$

Taking inverse Bayawa transform of (23) gives

$$S(v) = ß^{-1}\left\{\frac{v^6}{1-v^4}\right\}$$

Therefore, exact solution $S(x) = sinhx$ (24)

Example4 Consider the second kind linear volterra integro-differential equation

$$s''(x) = -1 + x + \int_0^x k(x-t)s(t)dt, s(0) = 1, s'(0) = 1 \tag{25}$$

Applying the Bayawa transform from both sides of (27) gives

$$ß\{s''(x)\} = ß\{-1\} + ß\{x\} + ß\left\{\int_0^x k(x-t)s(t)dt\right\} \tag{26}$$

Using convolution theorem of Bayawa transform (26) gives

$$\frac{1}{v^4}S(v) - s(0) - v^2 s'(0) = -v^4 + v^6 + \frac{1}{v^2}ß\{x\} * ß\{s(v)\}$$

$$\frac{1}{v^4}S(v) - s(0) - v^2 s'(0) = -v^4 + v^6 + \frac{1}{v^2} \cdot v^6 S(v) \tag{27}$$

Using initial condition s $s(0) = 1, s'(0) = 1$ equation above

$$\frac{1}{v^4}S(v) - 1 - v^2(1) = -v^4 + v^6 + v^4 S(v) \tag{28}$$

$$S(v)\left\{\frac{1}{v^4}-\ v^4\right\}=-v^4+v^6+1+v^2$$

$$S(v)=v^4\,\frac{\{-v^4+v^6+1+v^2\}}{1-v^8}$$

$$S(v)=\ v^4\,\frac{\left\{\{1-v^4\}+v^2\{1-v^4\}\right\}}{1-v^8} \tag{29}$$

Taking inverse Bayawa transform of (29) gives

$$S(v)=ß^{-1}\left\{\frac{v^6}{1+v^4}+\frac{v^4}{1+v^4}\right\}$$

Therefore, exact solution $S(x)=\ sinx\ +cosx$ (30)

Example5 Consider the second kind linear volterra integro-differential equation

$$s'''(x)=-1+\int_0^x s(t)dt, s(0)=s'(0)=1, s''(0)=-1 \tag{31}$$

Applying the Bayawa transform from both sides of (31), gives

$$ß\{s'''(x)\}=ß\{-1\}+\ ß\left\{\int_0^x s(t)dt\right\} \tag{32}$$

Using convolution theorem of Bayawa transform of (32), gives

$$\frac{1}{v^6}S(v)-\frac{1}{v^2}s(0)-s'(0)-v^2s''(0)=-v^4+\ \frac{1}{v^2}ß\{1\}*ß\{s(v)\}$$

$$\frac{1}{v^6}S(v)-\frac{1}{v^2}s(0)-s'(0)-v^2s''(0)=-v^4+\ \frac{1}{v^2}\cdot\ v^4S(v)$$

$$\frac{1}{v^6}S(v)-\frac{1}{v^2}s(0)-s'(0)\ -v^2s''(0)=-v^4+\quad v^2S(v) \tag{33}$$

Using initial condition s $s(0)=s'(0)=1, s''(0)=-1$ equation above, gives

$$\frac{1}{v^6}S(v)-\frac{1}{v^2}(1)-(1)\ -v^2(-1)=-v^4+\quad v^2S(v) \tag{34}$$

$$S(v)\left\{\frac{1}{v^6}-\ v^2\right\}=-v^4+\frac{1}{v^2}-\ v^2+1=\frac{-v^4+1-v^4+v^2}{v^2}$$

$$S(v)=v^4\,\frac{\{-v^4+v^6+1+v^2\}}{1-v^8}$$

$$S(v)=\ v^4\,\frac{\left\{\{1-v^4\}+v^2\{1-v^4\}\right\}}{1-v^8} \tag{35}$$

Taking inverse Bayawa transform of (35) gives

$$S(v)=ß^{-1}\left\{\frac{v^6}{1+v^4}+\frac{v^4}{1+v^4}\right\}$$

Therefore, exact solution $S(x) = sinx + cosx$ (36)

Conclusion

In this work, we proposed Bayawa transform method for solving second kind linear Volterra integro - differential equations for finding exact results. This method is a powerful procedure that can be use straight linear integral equations without taking too much time for calculation.

Declaration of Conflicting Interests

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