



Edge Trimagic Graceful Labeling of Tadpole Graph

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Abstract

A (p, q) graph G is called an edge trimagic graceful if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, p + q\}$ such that for each edge xy in $E(G)$, $|f(x) - f(xy) + f(y)| = C_1$ or C_2 or C_3 , where C_1, C_2 and C_3 are constants. In this paper, we proved that the Tadpole graphs are edge trimagic graceful graphs.

Keywords: Graph, trimagic, graceful, trimagic graceful, Tadpole graph.

AMS Subject Classification: 05C78

1. INTRODUCTION

Labeling of a graph G is an assignment of labels to either the vertices or the edges or both subject to certain conditions. Graph labeling was first introduced in 1960's. Graph labeling are of many types such as magic, bimagic, trimagic, antimagic, graceful, harmonious, equitable, etc. In this paper, we are going to study about trimagic graceful labeling of some ladder family graphs.

Magic labeling was introduced by Sedlacek [4]. In 1970, Kotzig and Rosa [3] defined, an edge magic labeling of graph G is a bijection $f: V \cup E \rightarrow \{1, 2, \dots, p + q\}$ such that, for each edge $uv \in E(G)$, $f(u) + f(uv) + f(v)$ is a magic constant. In 2004, Edge Bimagic labeling was introduced by J. B. Babujee [1] as a graph G with a bijection $f: V \cup E \rightarrow \{1, 2, \dots, p + q\}$ such that, for each edge $uv \in E(G)$, $f(u) + f(uv) + f(v)$ is either k_1 or k_2 .

In 2013, C. Jayasekaran, M. Regees and C. Davidraj [2] introduced the edge trimagic total labeling of graphs. An edge trimagic total labeling of a (p, q) graph G is a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, p + q\}$ such that for each edge $xy \in E(G)$, the value of $f(x) + f(xy) + f(y)$ is equal to any of the distinct constants k_1 or k_2 or k_3 . A (p, q) graph G is called an edge magic graceful if there exists a bijection $f: V(G) \cup E(G) \rightarrow \{1, 2,$

..., $p + q$ such that for each edge xy in $E(G)$ the value of $|f(x) + f(y) - f(xy)| = k$, a constant [6]. The graph G is said to be super edge magic graceful if $V(G) = \{1, 2, \dots, p\}$.

A closed walk in which no vertices, except the end vertices, are repeated is called the **cycle** and the number of edges in a cycle is called its length. An open walk in which no vertex appears more than once is called a **path**. A path with n vertices is denoted by P_n . A **tadpole** $T(n, m)$ is the graph obtained by appending a path P_m to a cycle C_n . We use the Dynamic Survey of Graph Labeling by Joseph A. Gallian [3] for more references. In this paper, we have proved that the Tadpole graph are edge trimagic graceful graphs.

2. MAIN RESULTS

Theorem 2.1: Every tadpole graph $T(n, m)$ is an edge trimagic graceful for all $n, m \geq 2$.

Proof: Let $V [T(n, m)] = \{u_i, v_j / 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertex set of the tadpole graph $T(n, m)$ and $E [T(n, m)] = \{ u_i u_{i+1}, v_j v_{j+1} / 1 \leq i \leq n, 1 \leq j \leq m \} \cup \{u_1 u_n\} \cup \{u_1 v_1\}$ be the edge set of the tadpole graph $T(n, m)$. Then the graph $T(n, m)$ has $n + m$ vertices and $n + m$ edges.

Case 1: n is odd & m is even

Define a bijection $\varphi: V \cup E \rightarrow \{1, 2, 3, \dots, 2(n + m)\}$ such that

$$\varphi(u_i) = \begin{cases} \frac{i+1}{2}, & 1 \leq i \leq n, i \text{ is odd} \\ \frac{n+i+1}{2}, & 1 \leq i \leq n, i \text{ is even} \end{cases}$$

$$\varphi(v_j) = \begin{cases} n + \frac{j+1}{2}, & 1 \leq j \leq m, j \text{ is odd} \\ n + \frac{m+j}{2}, & 1 \leq j \leq m, j \text{ is even} \end{cases}$$

$$\varphi(u) = 1; \varphi(uv_1) = 2n + m + 1$$

$$\varphi(uu_n) = n + m + 1; \varphi(u_i u_{i+1}) = n + m + i + 1, 1 \leq i \leq n - 1$$

$$\text{and } \varphi(v_j v_{j+1}) = n + m + j + 1, 1 \leq j \leq m - 1$$

Now we prove this labeling is edge trimagic graceful,

$$\begin{aligned} \text{Consider the edge } uu_n, \quad | \varphi(u) - \varphi(uu_n) + \varphi(u_n) | &= | 1 - (n + m + 1) + \frac{n+1}{2} | \\ &= | \frac{1-n-2m}{2} | = c_1 \text{ (say)} \end{aligned}$$

$$\begin{aligned} \text{Consider the edge } uv_1, \quad | \varphi(u) - \varphi(uv_1) + \varphi(v_1) | &= | 1 - (2n + m + 1) + n + 1 | \\ &= | 1 - n - m | = c_2 \text{ (say)} \end{aligned}$$

Consider the edge $u_i u_{i+1}, 1 \leq i \leq n - 1$

$$\begin{aligned} \text{For odd } i, \quad | \varphi(u_i) - \varphi(u_i u_{i+1}) + \varphi(u_{i+1}) | &= | \frac{i+1}{2} - (n + m + i + 1) + \frac{n+i+2}{2} | \\ &= | \frac{1-n-2m}{2} | = c_1 \end{aligned}$$

$$\begin{aligned} \text{For even } i, \quad | \varphi(u_i) - \varphi(u_i u_{i+1}) + \varphi(u_{i+1}) | &= | \frac{n+i+1}{2} - (n + m + i + 1) + \frac{i+2}{2} | \\ &= | \frac{1-n-2m}{2} | = c_1 \end{aligned}$$

Consider the edge $v_jv_{j+1}, 1 \leq j \leq m - 1$

$$\text{For odd } i, \left| \varphi(v_j) - \varphi(v_jv_{j+1}) + \varphi(v_{j+1}) \right| = \left| n + \frac{j+1}{2} - (2n + m + j + 1) + n + \frac{m+j+1}{2} \right| = \left| \frac{-m}{2} \right| = c_3 \text{ (say)}$$

$$\text{For even } i, \left| \varphi(v_j) - \varphi(v_jv_{j+1}) + \varphi(v_{j+1}) \right| = \left| n + \frac{m+j}{2} - (2n + m + j + 1) + n + \frac{j+2}{2} \right| = \left| \frac{-m}{2} \right| = c_3$$

Hence, for each edge $uv \in E [T(n, m)]$, $\left| \varphi(u) - \varphi(uv) + \varphi(v) \right|$ will form any one of the constants $c_1 = \left| \frac{1-n-2m}{2} \right|$, $c_2 = |1 - n - m|$ and $c_3 = \left| \frac{-m}{2} \right|$. Hence the Tadpole graph $T(n, m)$ is an edge trimagic graceful for n is odd & $m (\geq 2)$ is even.

Case 2: Both n & m are odd.

Define a bijection $\varphi: V \cup E \rightarrow \{1, 2, 3, \dots, 2(n + m)\}$ such that

$$\varphi(u_i) = \begin{cases} \frac{i+1}{2}, & 1 \leq i \leq n, i \text{ is odd} \\ \frac{n+i+1}{2}, & 1 \leq i \leq n, i \text{ is even} \end{cases}$$

$$\varphi(v_j) = \begin{cases} n + \frac{j+1}{2}, & 1 \leq j \leq m, i \text{ is odd} \\ n + \frac{m+j+1}{2}, & 1 \leq j \leq m, i \text{ is even} \end{cases}$$

$$\varphi(u) = 1; \quad \varphi(uv_1) = 2n + m + 1$$

$$\varphi(uu_n) = n + m + 1; \quad \varphi(u_iu_{i+1}) = n + m + i + 1, 1 \leq i \leq n - 1$$

$$\varphi(v_jv_{j+1}) = 2n + m + j + 1, 1 \leq j \leq m - 1$$

Now we prove this labeling is edge trimagic graceful,

Consider the edge uu_n ,

$$\left| \varphi(u) - \varphi(uu_n) + \varphi(u_n) \right| = \left| 1 - (n + m + 1) + \frac{n+1}{2} \right| = \left| \frac{1-n-2m}{2} \right| = c_1 \text{ (say)}$$

Consider the edge uv_1 ,

$$\left| \varphi(u) - \varphi(uv_1) + \varphi(v_1) \right| = \left| 1 - (2n + m + 1) + n + 1 \right| = \left| 1 - n - m \right| = c_2 \text{ (say)}$$

Consider the edge $u_iu_{i+1}, 1 \leq i \leq n - 1$

$$\text{For odd } i, \left| \varphi(u_i) - \varphi(u_iu_{i+1}) + \varphi(u_{i+1}) \right| = \left| \frac{i+1}{2} - (n + m + i + 1) + \frac{n+i+2}{2} \right| = \left| \frac{1-n-2m}{2} \right| = c_1$$

$$\text{For even } i, \left| \varphi(u_i) - \varphi(u_iu_{i+1}) + \varphi(u_{i+1}) \right| = \left| \frac{n+i+1}{2} - (n + m + i + 1) + \frac{i+2}{2} \right| = \left| \frac{1-n-2m}{2} \right| = c_1$$

Consider the edge $v_jv_{j+1}, 1 \leq j \leq m - 1$

For odd j , $|\varphi(v_j) - \varphi(v_j v_{j+1}) + \varphi(v_{j+1})| = \left| n + \frac{j+1}{2} - (2n + m + j + 1) + n + \frac{m+j+2}{2} \right| = \left| \frac{1-m}{2} \right| = c_3$ (say)

For even j , $|\varphi(v_j) - \varphi(v_j v_{j+1}) + \varphi(v_{j+1})| = \left| n + \frac{m+j+1}{2} - (2n + m + j + 1) + n + \frac{j+2}{2} \right| = \left| \frac{1-m}{2} \right| = c_3$

Hence, for each edge $uv \in E [T(n, m)]$, $|\varphi(u) - \varphi(uv) + \varphi(v)|$ will form any one of the constants $c_1 = \left| \frac{1-n-2m}{2} \right|$, $c_2 = |1 - n - m|$ and $c_3 = \left| \frac{1-m}{2} \right|$. Hence the Tadpole graph $T(n, m)$ is an edge trimagic graceful for both n & m are odd.

Case 3: n is even & m is odd

Define a bijection $\varphi: V \cup E \rightarrow \{1, 2, 3, \dots, 2(n + m)\}$ such that

$$\varphi(u_i) = \begin{cases} \frac{i+1}{2}, & 1 \leq i \leq n, \text{ i is odd} \\ \frac{n+i}{2}, & 1 \leq i \leq n, \text{ i is even} \end{cases}$$

$$\varphi(v_j) = \begin{cases} n + \frac{j+1}{2}, & 1 \leq j \leq m, \text{ i is odd} \\ n + \frac{m+j+1}{2}, & 1 \leq j \leq m, \text{ i is even} \end{cases}$$

$\varphi(u) = 1 ; \varphi(uv_1) = 2n + m + 1$

$\varphi(uu_n) = 2n + m ; \varphi(u_i u_{i+1}) = n + m + i, 1 \leq i \leq n - 1$

$\varphi(v_j v_{j+1}) = 2n + m + j + 1, 1 \leq j \leq m - 1$

Now we prove this labeling is edge trimagic graceful,

Consider the edge uu_n ,

$|\varphi(u) - \varphi(uu_n) + \varphi(u_n)| = |1 - (2n + m) + n| = |1 - n - m| = c_1$ (say)

Consider the edge $u_1 v_1$,

$|\varphi(u) - \varphi(uv_1) + \varphi(v_1)| = |1 - (2n + m + 1) + n + 1| = |1 - n - m| = c_1$

Consider the edge $u_i u_{i+1}, 1 \leq i \leq n - 1$

For odd i , $|\varphi(u_i) - \varphi(u_i u_{i+1}) + \varphi(u_{i+1})| = \left| \frac{i+1}{2} - (n + m + i) + \frac{n+i+1}{2} \right| = \left| \frac{2-n-2m}{2} \right| = c_2$ (say)

For even i , $|\varphi(u_i) - \varphi(u_i u_{i+1}) + \varphi(u_{i+1})| = \left| \frac{n+i}{2} - (n + m + i) + \frac{i+2}{2} \right| = \left| \frac{2-n-2m}{2} \right| = c_2$

Consider the edge $v_j v_{j+1}, 1 \leq j \leq m - 1$

For odd j , $|\varphi(v_j) - \varphi(v_j v_{j+1}) + \varphi(v_{j+1})| = \left| n + \frac{j+1}{2} - (2n + m + j + 1) + n + \frac{m+j+2}{2} \right| = \left| \frac{1-m}{2} \right| = c_3$ (say)

For even j , $\left| \varphi(v_j) - \varphi(v_j v_{j+1}) + \varphi(v_{j+1}) \right| = \left| n + \frac{m+j+1}{2} - (2n + m + j + 1) + n \right.$
 $\left. + \frac{j+2}{2} \right| = \left| \frac{1-m}{2} \right| = c_3$

Hence, for each edge $uv \in E [T(n, m)]$, $\left| \varphi(u) - \varphi(uv) + \varphi(v) \right|$ will form any one of the constants $c_1 = |1 - n - m|$, $c_2 = \left| \frac{2-n-2m}{2} \right|$ and $c_3 = \left| \frac{1-m}{2} \right|$. Hence the Tadpole graph $T(n, m)$ is an edge trimagic graceful for n is even & m is odd.

Case 4: Both n & m are even.

Define a bijection $\varphi: V \cup E \rightarrow \{1, 2, 3, \dots, 2(n + m)\}$ such that

$$\varphi(u_i) = \begin{cases} \frac{i+1}{2}, 1 \leq i \leq n, i \text{ is odd} \\ \frac{n+i}{2}, 1 \leq i \leq n, i \text{ is even} \end{cases}$$

$$\varphi(v_j) = \begin{cases} n + \frac{j+1}{2}, 1 \leq j \leq m, i \text{ is odd} \\ n + \frac{m+j}{2}, 1 \leq j \leq m, i \text{ is even} \end{cases}$$

$\varphi(u) = 1 ; \varphi(uv_1) = 2n + m + 1$

$\varphi(uu_n) = 2n + m ; \varphi(u_i u_{i+1}) = n + m + i, 1 \leq i \leq n - 1$

and $\varphi(v_j v_{j+1}) = 2n + m + j + 1, 1 \leq j \leq m - 1$

Now we prove this labeling is edge trimagic graceful,

Consider the edge uu_n ,

$\left| \varphi(u) - \varphi(uu_n) + \varphi(u_n) \right| = \left| 1 - (2n + m + 1) + n \right| = \left| 1 - n - m \right| = c_1$ (say)

Consider the edge uv_1 ,

$\left| \varphi(u) - \varphi(uv_1) + \varphi(v_1) \right| = \left| 1 - (2n + m + 1) + n + 1 \right| = \left| 1 - n - m \right| = c_1$

Consider the edge $u_i u_{i+1}, 1 \leq i \leq n - 1$

For odd i , $\left| \varphi(u_i) - \varphi(u_i u_{i+1}) + \varphi(u_{i+1}) \right| = \left| \frac{i+1}{2} - (n + m + i) + \frac{n+i+1}{2} \right|$
 $= \left| \frac{2-n-2m}{2} \right| = c_2$ (say)

For even i , $\left| \varphi(u_i) - \varphi(u_i u_{i+1}) + \varphi(u_{i+1}) \right| = \left| \frac{n+i}{2} - (n + m + i) + \frac{i+2}{2} \right|$
 $= \left| \frac{2-n-2m}{2} \right| = c_2$

Consider the edge $v_j v_{j+1}, 1 \leq j \leq m - 1$

For odd i , $\left| \varphi(v_j) - \varphi(v_j v_{j+1}) + \varphi(v_{j+1}) \right| = \left| n + \frac{j+1}{2} - (2n + m + j + 1) + n \right.$
 $\left. + \frac{m+j+1}{2} \right| = \left| \frac{-m}{2} \right| = c_3$ (say)

For even i , $\left| \varphi(v_j) - \varphi(v_j v_{j+1}) + \varphi(v_{j+1}) \right| = \left| n + \frac{m+i}{2} - (2n + m + i + 1) + n \right.$
 $\left. + \frac{i+2}{2} \right| = \left| \frac{-m}{2} \right| = c_3$

Hence, for each edge $uv \in E [T(n, m)]$, $|\varphi(u) - \varphi(uv) + \varphi(v)|$ will form any one of the constants $c_1 = |1 - n - m|$, $c_2 = \left\lfloor \frac{2-n-2m}{2} \right\rfloor$ and $c_3 = \left\lfloor \frac{-m}{2} \right\rfloor$. Hence the Tadpole graph $T(n, m)$ is an edge trimagic graceful for both n & m are even.

Hence, the theorem follows from the above four cases.

Corollary 2.2: The Tadpole $T(n, m)$ admits a super edge trimagic graceful labeling for all $n, m \geq 2$.

Proof: We proved the Tadpole $T(n, m)$ admits an edge trimagic graceful labeling. The labeling given in the proof of theorem 2.1, the vertices get labels $1, 2, \dots, n + m$. Since the Tadpole graph $T(n, m)$ has $n + m$ vertices and these $n + m$ vertices have labels $1, 2, \dots, n + m$ for all n and $m \geq 2$ the Tadpole graph $T(n, m)$ is a super edge trimagic graceful graph.

Example 2.3: A super edge trimagic graceful labeling of the Tadpole $T(9, 11)$ is given in figure 1.

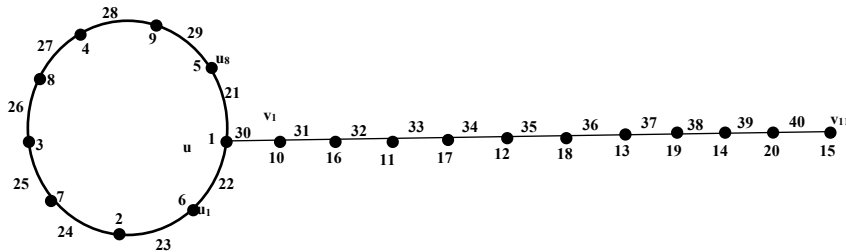


Figure 1: $T(9, 11)$ with $c_1 = 15, c_2 = 19$ and $c_3 = 5$

3. CONCLUSION

In this paper we have discussed about the edge trimagic graceful labeling and super edge trimagic graceful labeling of tadpole Graphs $T(n, m)$ with for all $n, m \geq 2$. We contribute some results to the theory of these edge trimagic graceful labeling. An example was provided at the end of the theorem for better understanding of the labeling pattern defined.

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The authors declare no potential conflicts of interest with respect to the research, authorship and publication of this article.

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